EXTERIOR DIFFERENTIAL SYSTEMS - ANY (REAL ANALYTIC) RIEMANNIAN SURFACE ISOMETRICALLY EMBEDS IN $\mathbb{R}^3$

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1. Definitions

An Exterior Differential System (EDS) is a pair $(M, \mathcal{I})$ with M a smooth manifold and differential ideal $\mathcal{I} \subseteq \Omega^*(M)$. By differential ideal I mean a graded ideal in the ring $\Omega^*(M)$ of differential forms on M which is furthermore closed under exterior differentiation.

Given such an $(M, \mathcal{I})$, we are usually interested in finding integral submanifolds, $i\colon N \hookrightarrow M$ so that $i^*\phi = 0$ for all $\phi \in \mathcal{I}$. In other words, submanifolds so that $\mathcal{I}|_{TN} = \langle 0 \rangle$.[1] The solutions to this problem will, for appropriately constructed $(M, \mathcal{I})$, correspond to solutions to interesting PDE or geometric problems.

Conveniently, there are very powerful tools developed in EDS which turn this existence problem into one of linear algebra and counting dimensions.

2. How to go from a PDE to an EDS

Example. Consider the ODE

$$\frac{dx}{dt} = F(t, x, y)$$
$$\frac{dy}{dt} = G(t, x, y)$$

with F, G smooth functions on $\mathbb{R}^3$. We can represent a solution $i\colon (a, b) \to \mathbb{R}^2$ to this system as an appropriate curve in $\mathbb{R} \times \mathbb{R}^2$, namely the graph of i. Conversely, we would like to develop an EDS whose integral curves are graphs of solutions. From the first equation we must have[2] $dx = F(t, x, y)dt$ and similarly from the second we have $dy = G(t, x, y)dt$ on such a curve. This suggests we consider the EDS $(\mathbb{R}^3, \mathcal{I})$ where $\mathcal{I} = \langle dx - F dt, dy - G dt \rangle$.

Indeed, an integral curve $S \subset \mathbb{R}^3$ will have tangent space at each point spanned by a vector $a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x} + c\frac{\partial}{\partial y}$, and the requrement that $\mathcal{I}$ restrict to zero on this space shows that the vector is parallel to $\frac{\partial}{\partial t} + F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y}$, which says exactly that S is in fact a solution to the ODE.

Example. Now consider the PDE

$$z_x = F(x, y, z)$$
$$z_y = G(x, y, z).$$

This can be modeled by $(\mathbb{R}^3, \mathcal{I})$ where $\mathcal{I} = \langle dz - F dx - G dy \rangle$. It is not hard to see[3] that an integral surface can be written (locally) as a graph $z = u(x, y)$. If we consider the

[1] This is why we ask that $\mathcal{I}$ be closed under differentiation: if $\phi|_{TN} = 0$ then so must $d\phi|_{TN} = 0$

[2] Here I should more accurately say, for example, $i^*dx = F(t, x(t), y(t))i^*dt$, but in practice it should be clear when a form is restricted to a submanifold.

[3] Indeed, we have $dx \wedge dy \neq 0$, for otherwise the condition $dz = F dx + G dy$ tells us that all 3 of dx, dy and dz are pairwise dependent.

tangent plane of this graph, spanned by $\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial z}$ and $\frac{\partial}{\partial y} + u_y \frac{\partial}{\partial z}$, we see that

$$u_x(x,y) = F(x,y,u(x,y))$$
$$u_y(x,y) = G(x,y,u(x,y))$$

so that integral surfaces correspond exactly to solutions.

3. This is cool because?

Rephrasing a question is only useful if it opens up new methods of solution. There are very powerful techniques, but for now let's see how we can use one of the basic tools of geometry, the Frobenius Theorem.

Theorem (Frobenius, EDS version). *Let $(M, \mathcal{I})$ be an EDS so that $\mathcal{I} = \langle \mathcal{I}^1 \rangle_{alg}$ ("$\mathcal{I}$ is generated **algebraically** by 1 forms") and $\dim \mathcal{I}^1|_{T_pM}$ is a constant r independent of each $p \in M$. Then each point of M has local coordinates $\mathbf{x} = (x^1, \ldots, x^{n+r})$ on U so that $\mathcal{I}|_U = \langle dx^{n+1}, \ldots, dx^{n+r} \rangle$.*

This is the same as the standard Frobenius theorem, and says that the n-dimensional integral manifolds in these coordinates (locally) are just those with $x^{n+1} = c^1, \ldots, x^{n+r} = c^r$ for some constants.

Example. We can immediately apply this to the PDE above, $(\mathbb{R}^3, \mathcal{I})$ with $\mathcal{I} = \langle dz - Fdx - Gdy \rangle$. In this case the *differential* ideal $\mathcal{I}$ is generated *algebraically* by $\zeta = dz - Fdx - Gdy$ exactly if $\zeta \wedge d\zeta = 0$. But it is straightforward to calculate that

$$\zeta \wedge d\zeta = (F_y - G_x + GF_z - FG_z) dx \wedge dy \wedge dz.$$

So, by Frobenious, if $(F_y - G_x + GF_z - FG_z) = 0$ then for each (x_0, y_0, z_0) there are coordinates so that the integral surface $z = z_0$ is the unique solution through (x_0, y_0, z_0).

4. Heavy machinery

To deal with more difficult problems it is necessary to develop some machinery. The basic strategy is to look at 'local' solutions to our EDS and ask if they glue together to give an honest solution.

Given an EDS $(E, \mathcal{I})$, an n-dimensional subspace $E \subset T_xM$ is an *integral element* of $\mathcal{I}$ if $\phi|_E = 0$ for each $\phi \in \mathcal{I}$. For fixed n we will call the set of these $V_n(\mathcal{I})$, which is a closed subset of the Grassmanian bundle $G_n(TM)$ over M.

To construct integral manifolds we often start with an integral manifold of dimension n and then try to extend it to an integral manifold of dimension $n+1$. To facilitate this,

Definition. For an integral element $E \in V_k(\mathcal{I})$ and a basis $e_1, \ldots, e_k$ of $E \subset T_xM$ we define the polar space (of extensions) by

$$H(E) = \{v \in T_x : \eta(v, e_1, \ldots, e_k) = 0, \forall \eta \in \mathcal{I} \cap \Omega^{k+1}(M)\}$$

Notice that finding the polar space at a point is a matter of linear algebra. In fact, often all that matters is the dimension of these spaces. Given an $E \in V_k(\mathcal{I})$, an integral element in T_xM, define the integer $c(E) = \dim(T_xM) - \dim(H(E))$. We have

Theorem (Cartan's Test). *Consider a real analytic EDS $(M, \mathcal{I})$ and an integral flag $(0) = E_0 \subset E_1 \subset \ldots \subset E_n$ so that $E_i \in V_i(\mathcal{I})$. If $V_n(\mathcal{I})$ is a smooth manifold in a neighborhood of E_n, of codimension*

$$c(E_0) + c(E_1) + \ldots + c(E_{n-1})$$

then there is a real analytic n-dimensional integral manifold $P \subset M$ passing through x and so that $T_xP = E_n$.

5. The orthonormal frame bundle on $\mathbb{R}^3$

To set up my final example, I want to make a quick digression to explain the (special) orthonormal frame bundle $\mathcal{F}$ on $\mathbb{R}^3$. In short, this is the bundle over $\mathbb{R}^3$ with fiber at each point p given by the set of orthonormal frames in $T_p\mathbb{R}^3$ which agree with the orientation of $\mathbb{R}^3$, which is to say that a point in $\mathcal{F}$ takes the form $(\mathbf{x}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ with the $\mathbf{e_i}$ a positively oriented orthonormal basis of $T_x\mathbb{R}^3$. To see what this looks like, notice that $SO(3,\mathbb{R})$ acts freely and transitively on each fiber, so $\mathcal{F}_p$ is diffeomorphic to $SO(3)$. In fact it is not hard to see that $\mathcal{F}$ is naturally isomorphic to

$$ASO(3) = \left\{ A \in GL(4,\mathbb{R})\colon A = \begin{pmatrix} 1 & 0 \\ \mathbf{t} & R \end{pmatrix},\ \mathbf{t} \in \mathbb{R}^3,\ R \in SO(3) \right\}.$$

To see this, we simply identify the point $(\mathbf{x}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}) \in \mathcal{F}$ with the element

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \end{pmatrix} \in ASO(3)$$

and vice versa.

The group $ASO(3)$ is a semidirect product of $\mathbb{R}^3$ and $SO(3)$, so we see that $\mathbb{R}^3$ is the homogeneous space $ASO(3)/SO(3)$. The fact that we can identify the orthonormal frame bundle with a Lie group is fantastic news because it gives us a canonical basis for the cotangent bundle via the *Maurer-Cartan form.*

Definition. For a Lie group G with Lie algebra $\mathfrak{g}$ there is a unique left-invariant $\mathfrak{g}$ valued 1-form on G whose restriction to $T_eG (= \mathfrak{g})$ is the identity[4]. This form is called the Maurer-Cartan form of G.

Exercise. For a matrix Lie group with embedding $g\colon G \to GL(n)$ the Maurer-Cartan form is simply $g^{-1}dg$

Example. For the group $ASO(3)$ the Maurer-Cartan form $\omega = g^{-1}dg$ takes the form

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & \omega_2^1 & \omega_3^1 \\ \omega^2 & -\omega_2^1 & 0 & \omega_3^2 \\ \omega^3 & -\omega_3^1 & -\omega_3^2 & 0 \end{pmatrix} \in \mathbb{R}^3 \oplus \mathfrak{so}(3).$$

By solving for dg we have

$$d\begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & \omega_2^1 & \omega_3^1 \\ \omega^2 & -\omega_2^1 & 0 & \omega_3^2 \\ \omega^3 & -\omega_3^1 & -\omega_3^2 & 0 \end{pmatrix}$$

[4] This may seem like a mysterious definition if you have not seen it before, but the crux of it is that we have an obvious identification between the tangent space at the identity and the vector space $\mathfrak{g}$. Furthermore, we can push this all around G by left translation to get identifications between each tangent space of G and $\mathfrak{g}$. Incidentally, this shows that the tangent bundle of any Lie group is the trivial bundle.

so that for example $d\mathbf{x} = \mathbf{e_1}\omega^1 + \mathbf{e_2}\omega^2 + \mathbf{e_3}\omega^3$ and $d\mathbf{e_3} = \mathbf{e_1}\omega^1_3 + \mathbf{e_2}\omega^2_3$. Even better, by taking the derivative of $\omega = g^{-1}dg$ we see[5] that $d\omega = -\omega \wedge \omega$, or in our case

$$d\begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & \omega^1_2 & \omega^1_3 \\ \omega^2 & -\omega^1_2 & 0 & \omega^2_3 \\ \omega^3 & -\omega^1_3 & -\omega^2_3 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & \omega^1_2 & \omega^1_3 \\ \omega^2 & -\omega^1_2 & 0 & \omega^2_3 \\ \omega^3 & -\omega^1_3 & -\omega^2_3 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & \omega^1_2 & \omega^1_3 \\ \omega^2 & -\omega^1_2 & 0 & \omega^2_3 \\ \omega^3 & -\omega^1_3 & -\omega^2_3 & 0 \end{pmatrix}.$$

As mentioned above, the ω^i, ω^i_j are a basis of the 1-forms on $ASO(3)$.

6. Isometric surfaces in $\mathbb{R}^3$

Suppose we have a Riemannian surface S. We fix at each point p an orthonormal framing $\mathbf{v_1}, \mathbf{v_2} \in T_pS$ and for later convenience call its dual coframing η^1, η^2. Now, for an isometric embedding $f\colon S \to \mathbb{R}^3$ we have a natural lift

$$\begin{aligned} \hat{f}\colon S &\to ASO(3) \\ p &\mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ f(p) & f'(\mathbf{v_1}) & f'(\mathbf{v_2}) & f'(\mathbf{v_1}) \times f'(\mathbf{v_2}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathbf{x} & \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \end{pmatrix}. \end{aligned}$$

Notice that this lift is chosen so that $\mathbf{e_1}$ and $\mathbf{e_2}$ span the tangent plane at each point of S and $\mathbf{e_3}$ is the normal to S. As a consequence we have

$$e_1\eta^1 + e_2\eta^2 = df = f^*dx = e_1 f^*\omega^1 + e_2 f^*\omega^2 + e_3 f^*\omega^3$$

which implies that $\eta^1 = f^*\omega^1$, $\eta^2 = f^*\omega^2$ and $f^*\omega^3 = 0$. Conversely, if we have a map $\hat{f}\colon S \to ASO(3)$ so that $f^*\omega^1 \wedge f^*\omega^2 \neq 0$,[6] and $0 = \eta^1 - f^*\omega^1 = \eta^2 - f^*\omega^2 = f^*\omega^3$ then its projection to $\mathbb{R}^3$ will be an isometric embedding. In other words, solutions to the exterior differential system $(M = S \times ASO(3), \mathcal{I} = \langle \omega^3, \eta^1 - \omega^1, \eta^2 - \omega^2 \rangle)$ with $\eta^1 \wedge \eta^2 \neq 0$ will be the graphs of isometric immersions of S into $\mathbb{R}^3$. So the question is, does this have solutions? We will use Cartan's test to answer this.

To answer this it will be necessary to find a set of forms which generate $\mathcal{I}$ algebraically. Recall that for an orthonormal coframing η^1, η^2 there is a form η^1_2 so that $d\eta^1 = -\eta^1_2 \wedge \eta^2$ and $d\eta^2 = \eta^1_2 \wedge \eta^1$. This form also satisfies $d\eta^1_2 = K\eta^1 \wedge \eta^2$ where K is the curvature of the metric.

From this we see that

$$d(\eta^1 - \omega^1) = -\eta^1_2 \wedge \eta^2 + \omega^1_2 \wedge \omega^2 \equiv (\omega^1_2 - \eta^1_2) \wedge \omega^2 \mod \mathcal{I}$$

and

$$d(\eta^2 - \omega^2) = \eta^1_2 \wedge \eta^1 - \omega^1_2 \wedge \omega^1 \equiv -(\omega^1_2 - \eta^1_2) \wedge \omega^1 \mod \mathcal{I}$$

are in $\mathcal{I}$. However, we have assumed that ω^1 and ω^2 form a basis of the cotangent bundle of an integral surface, so solutions to $\mathcal{I}$ will also have $\omega^1_2 - \eta^1_2 = 0$. In light of this, we consider the new EDS $(M, \mathcal{I}' = \langle \omega^3, \eta^1 - \omega^1, \eta^2 - \omega^2, \omega^1_2 - \eta^1_2 \rangle)$, whose solutions also correspond to isometric embeddings of S.[7]

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[5] Hint: $d(g^{-1}) = -g^{-1}dg\, g^{-1}$.

[6] That is to say, the image of S is transverse to the fiber, so that $\pi \circ f$ is an immersion.

[7] It is not obvious from what I have said why it is necessary to expand our ideal. The original system $(M, \mathcal{I})$ is not *involutive*, which to us means that Cartan's test would have failed to guarantee us integral surfaces. Our new ideal $\mathcal{I}'$ satisfies the hypothesis of Cartan's test, so we see that $\mathcal{I}'$ and thus $\mathcal{I}$ does indeed have integral surfaces. This is a general phenomenon, wherein we can *prolong* a system to get a new one which shares the same solutions but is furthermore involutive.

Our new ideal $\mathcal{I}'$ is generated algebraically by

$$\theta_0 = \omega^3 \tag{1}$$

$$\theta_1 = \omega^1 - \eta^1 \tag{2}$$

$$\theta_2 = \omega^2 - \eta^2 \tag{3}$$

$$\theta_3 = \omega^1_2 - \eta^1_2 \tag{4}$$

$$d\theta_0 = \omega^1_3 \wedge \omega^1 + \omega^2_3 \wedge \omega^2 \tag{5}$$

$$d\theta_3 = \omega^1_3 \wedge \omega^2_3 - K\omega^1 \wedge \omega^2 \tag{6}$$

To find the codimension of $V_2(\mathcal{I}')$ in $G_2(TM)$ we parameterize the open set of planes in T_pM with $\eta^1 \wedge \eta^2 \neq 0$ by the real numbers p^a_i, h_{ij}, where

$$\omega^a = p^a_1\eta^1 + p^a_2\eta^2 \tag{7}$$

and

$$\omega^1_2 = p^4_1\eta^1 + p^4_2\eta^2 \tag{8}$$

$$\omega^1_3 = h_{11}\eta^1 + h_{12}\eta^2 \tag{9}$$

$$\omega^2_3 = h_{21}\eta^1 + h_{22}\eta^2. \tag{10}$$

The equations (1)-(4) then determine all 8 of the p^a_i and (5) and (6) give 2 independent relations between the remainder,[8] so that $V_2(\mathcal{I}')$ has codimension 10.

On the other hand, given an integral element $E \in V_2(\mathcal{I}')$ defined by the equations (7) - (10) let us fix an integral flag $(0) \subset E_1 \subset E$ where E_1 is spanned by an element e. It follows from the definition that $H(E_0)$ has codimension equal to the number of independent 1 forms in $\mathcal{I}'$, so regardless of our choice of flag, $c_0 = 4$. The space $H(E_1)$ is the set of vectors v which annihilate the four 1-forms θ_i,[9] as well as the 1-forms

$$\begin{aligned} i_e(\omega^1_3 \wedge \omega^1 + \omega^2_3 \wedge \omega^2) &= \omega^1_3(e)\omega^1 + \omega^2_3(e)\omega^2 - \omega^1(e)\omega^1_3 - \omega^2(e)\omega^2_3 \\ i_e(\omega^1_3 \wedge \omega^2_3 - K\omega^1 \wedge \omega^2) &= K\omega^2(e)\omega^1 - K\omega^1(e)\omega^2 - \omega^2_3(e)\omega^1_3 + \omega^1_3(e)\omega^2_3. \end{aligned}$$

Provided that either $h_{12} \neq 0$ or $h_{11} + h_{22} \neq 0$ these two forms will be independent of each other and the θ_i, so $H(E_1)$ has codimension 6. In this case $c_0 + c_1 = 4 + 6 = 10 = \operatorname{codim}(V_2(\mathcal{I}'))$, so we see by Cartan's test that there is an integral manifold to $\mathcal{I}'$ tangent to E. However, we can always find an integral element E where $h_{12} \neq 0$ or $h_{11} + h_{22} \neq 0$, and then this will hold in a neighborhood. This concludes the proof that any real analytic Riemannian surface can be isometrically embedded in $\mathbb{R}^3$.

[8] In fact, (5) is $h_{12} = h_{21}$, reflecting the fact that the matrix (h_{ij}) is actually the second fundamental form. The equation (6) is $h_{11}h_{22} - h^2_{12} = K$, the Gauss equation.

[9] The astute observer may ask why the 1-forms matter since the definition of $H(E_1)$ only depends on the 2-forms in $\mathcal{I}'$. The answer is that $\mathcal{I}'$ is an ideal. For example, if α is a one form on which $\alpha(e) = 1$ then $v \in H(E_1)$ requires $\theta \wedge \alpha(v, e) = \theta(v)\alpha(e) - \theta(e)\alpha(v) = \theta(v)$, where we use the fact that $\theta(e) = 0$.